Cause-and-effect relationships prove the impossibility of the existence of closed numerical cycles in the Collatz conjecture

Azamat Bayanov
(S.Seifullin Kazakh Agro Technical Research University, Astana, Kazakhstan)

Abstract

The famous L. Collatz problem 3x+1 needs no introduction. This article used the method of proof by contradiction regarding the possible existence of cycles other than the 4–2–1 cycle. The work provides evidence in favor of the fact that closed numerical cycles in the Collatz sequence cannot exist. The results discussed are demonstrated using some examples.

Keywords

Collatz conjecture, 3x+1 problem, algorithm of Collatz, number sequence.

Introduction

The Collatz conjecture is one of the most famous unsolved problems in mathematics [1, 2]. The Collatz conjecture can be formulated as follows: take any positive integer x, if x is an even number, divide it by 2 to get x/2, if x is odd, multiply it by 3 and add 1 to get $3 \times x + 1$. Repeat the process again and again. The Collatz conjecture is that whatever number (i.e. x) is chosen, the process will always eventually reach 1 [3].

Preliminaries

All natural numbers are divided into 2 groups: even and odd. In turn, directly for this proof, we divide the even numbers into 2 more groups:

- numbers that can be obtained by transforming 3x+1 from the previous number in the sequence (starting with the number 4 – all numbers with a period of 6 and to infinity). Further, only within the framework of this proof, we will call them universal;
- all other even numbers. Further, only within the framework of this proof, we will call them prime even numbers.

Note that universal numbers can be obtained either by transforming the previous number into 3x+1, or by dividing the previous number by 2. As a result, we have 3 main groups of numbers that cover the entire infinite set of natural numbers:

- odd (appears in the sequence only as a result of dividing the previous number by 2);
- prime even numbers (appear in the sequence only as a result of dividing the previous number by 2);
- universal (appears in the sequence both as a result of dividing the previous number by 2, and as a result of transforming the previous number into 3x+1).

By the way, in the only currently known cycle 4–2–1, all 3 groups described above are present.

Suppose that among the entire infinite set of natural numbers there is a sequence *(in this case the sequence aa'* in Figure 1) that tends to rise and fall (according to the conditions of Collatz conjecture) an arbitrarily infinite number of times, but in the end returns to a number that has already been in this sequence before (except for the 4–2–1 cycle.). This, in turn, would mean that there is a certain cycle, hypothetical for now, which would prove the opposite to the fact that not all natural numbers sooner or later come to a point with the value 1.

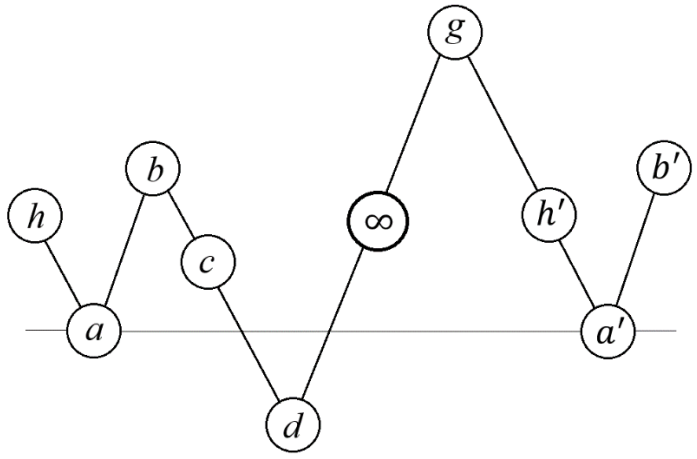

Figure 1 – Hypothetical closed numerical cycles.

Note that for a hypothetical cycle, the region d–g in Figure 1 can include an arbitrarily infinite number of numbers.

For convenience of narration and consistency with the essence, the number that occurs first for the second time in a given sequence and closes the cycle *(in this case the sequence aa')* will be called a turning number. It does not require proof that in any possible hypothetical cycle there can be only one turning number, which is the first to close the cycle in the sequence under consideration, otherwise – if the second were there ahead of the first, the cycle would close one step earlier and it would be this number that would have to be considered in as a turning.

We will consider it an axiom that the nearest neighbors of the turning number are different numbers, because it is always obtained from one number and transformed into another.

If in the sequence *(in this case the sequence aa')* the estimated turning number reaches a' (that is, essentially itself) after passing through section d–g, then the graph is clearly true to the further narration.

If the supposed turning number is encountered earlier *(for example in the interval of section d–g)*, then this does not fundamentally change anything, because it would also be fair to consider this situation in the context of this example.

If, hypothetically, starting from the number *a*, after a certain number of numbers in the sequence some other number appears twice, for example *d*, then this also does not fundamentally change anything, since we can further consider it in the same way as the number *a* presented here and below.

Based on the 3 statements above, it becomes obvious that in any of the hypothetically possible cycles there is one and only one turning number, which will appear first in the sequence of numbers under study.

Previously, all natural numbers were divided into 3 groups: odd (comprising ½ of the total set of natural numbers); universal (every 6th number starting from the number 4, make up ~ 1/6 of the entire set of natural numbers); prime even numbers (they make up the remaining ~ 1/3 of the set of natural numbers). Obviously, the only possible turning number in any of the hypothetical cycle belongs to one of three groups. Let us determine which group or groups it may belong to, acting as a turning number for the formation of a closed numerical cycles.

Odd numbers

Let us assume that the initial supposed turning number *a* (Fig. 1) is an odd number, and, after an arbitrarily large amount of transformation, there remains only one action left to perform *(x/2 or 3× x+1)* to obtain the turning number in the sequence in the second times, which will naturally mark a closed numerical cycle.

The fact does not require proof that since this number is odd, then in absolutely any sequence, however infinitely long or short, it can only be obtained by dividing the previous number in half. Let us denote this only possible path by the section $h'a'$. Since this only possible path $h'a'$ to a' is valid, then the only possible path *ha* leading to *a* is also valid, since a and a' are essentially the same number and are spaced in different places on the graph sides only for convenience of presentation, and is highlighted a second time to make it clear that at this point the hypothetical cycle will begin a second procession.

It is obvious that a number *(in this case an even prime or universal, expressed by h')* preceding a hypothetical turning number *(in this case an odd one, expressed by a')* cannot be present in a hypothetical closed numerical cycles *(in this case in the cycle aa')*, since otherwise the hypothetical closed numerical cycles would close in it *(in this case the cycle hh')* and would appear as the desired hypothetical closed numerical cycles when moving through the sequence earlier than the cycle aa'). It would be fair to extrapolate this result to all odd numbers from the natural set and draw a simple but irrefutable conclusion that closed numerical cycles with odd numbers as a turning point are impossible, since a hypothetical cycle, when moving through a sequence, will always close a little earlier than it – on the previous even number.

I would like to give several examples of this statement, but, for obvious reasons, I will give only 1 (this example is not exhaustive, but by the end of the article its evidentiary ability will become obvious) (Fig. 2).

This example demonstrates that the number 8, after forming the turning number 4, will never again fall into the 4–2–1 cycle. The opposite would mean that if the number 8 were to fall into the cycle 4–2–1, then when moving through the sequence, it would be the number 8 that would form the cycle one step before the number 4 would do so. Obviously, this is true both for this single existing example, and for other hypothetical closed numerical cycles if they exist.

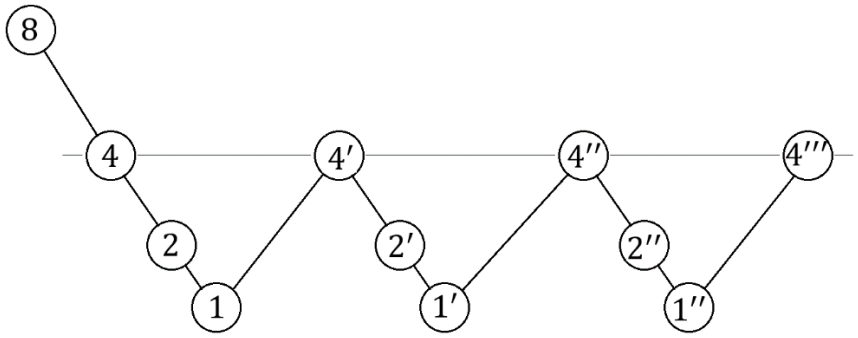

Figure 2 – Closed numerical cycles.

Next, we will consider the remaining 2 groups of numbers as turning ones.

Prime even numbers

It would be fair if we skip the proof of the impossibility of the existence of closed numerical cycles with a prime even number as a turning number, since this group of numbers in any sequence is formed like odd numbers only by dividing the preceding number by 2, and, accordingly, has an identical conclusion to the previous one – a closed numerical cycles with a prime even number as a turning number cannot exist.

Universal numbers

As for the above-described considerations, for a universal number as a turning point in a hypothetical cycle, the condition will be valid: after universal number formation by a previous number, the previous number itself can no longer appear in the hypothetical closed numerical cycles, otherwise it would itself form its own cycle one step forward universal. Note that universal numbers, unlike the previous 2 groups, in any sequence can be obtained either by converting the previous odd number into 3x+1, or by dividing the previous prime even number by 2, which makes them special in relation to the abovementioned groups (Fig. 3).

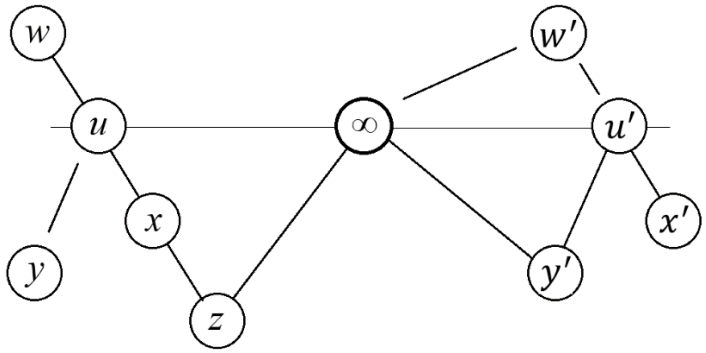

Figure 2 – Hypothetical closed numerical cycles.

Let us denote the universal number as a turning number in the last hypothetically possible cycle as u. Let us assume that, as in previous cases, the number u will undergo an arbitrarily large or infinitely small number of transformations (according to the conditions of the sequence), and, ultimately, will close on itself in u'. Let us recall that u and u' are essentially the same number, and are spaced apart on the graph only for ease of presentation, and highlighted a second time with a prime to make it clear that at this point the hypothetical cycle will begin the second circle of movement. Let us also repeat that there are no other repeating numbers between u and u', otherwise they would be the subject of consideration.

From the previous statement that the universal numbers can be obtained either by converting the preceding odd number to 3x+1 or by dividing the preceding prime even number by 2, it follows that the universal number denoted u, acting as a turning point in the last possible hypothetical cycle can be obtained either by transforming the odd number preceding it *(in this case y)* into 3x+1, or from the prime even number preceding it *(in this case w)* by dividing by 2. The twin brother of $u - u'$ is formed from y' and w' in a similar way.

As in the earlier cases, it is obvious that a number *(a prime even, since no other precedes the universal, denoted w)* preceding a turning number in a hypothetical cycle *(in this case, a universal, denoted u)* cannot be present in a hypothetical cycle (in this particular case uu'), since otherwise the hypothetical cycle would close in it *(in this case ww')* and would be a hypothetical cycle that would be worth considering earlier than the current one (uu').

It is also obvious that a number *(in this case, odd, expressed by y)* preceding a turning number in a hypothetical cycle *(in this case, a universal, denoted u)* cannot be present in a hypothetical cycle (in this particular case uu'), since otherwise the hypothetical cycle would close in it *(in this case yy')* and would be a hypothetical cycle that would be worth considering earlier than the current one (uu').

It is fair to note that with universal numbers, in contrast to prime evens and odd ones, the situation is somewhat different. The whole point is that we can obtain the turning number of a hypothetical cycle in the form of a universal one *(in this particular case, denoted by u)* by transforming it from the previous odd one *(in this case, through 3y+1)*, and again access it so that it closes a hypothetically assumed cycle (uu'), by dividing by 2 the number preceding it *(in this particular case $w'/2$)*.

And the second way is possible, namely, we obtain the turning number of the hypothetical cycle in the form of a universal one *(in this particular case, designated u)* by dividing by 2 the number preceding it *(in this particular case, wu)*, and again reach it so that it closes the hypothetically assumed cycle (uu'), by converting it from the preceding odd one (in this particular case by $3y'$+1).

According to the last two paragraphs, it turns out that the turning number in the form of a universal, obtained strictly according to the conditions of the sequence in one of two possible ways, has the ability, through an arbitrarily short or infinitely long number of operations, to reach itself as a result of the same mathematical operations *($3 \times x + 1$, $x/2$)* also in two possible ways. But this is an obvious contradiction, since the sequence can only move along one single path. This means

that a closed numerical cycles with a universal number as a turning number cannot exist either.

Conclusions

None of the 3 groups considered, which cover absolutely the entire set of natural numbers, contains a number that can close any sequence without violating the elementary logical rules of the existence of a closed numerical cycle. This, in turn, suggests that in the entire set of natural numbers, closed numerical cycles other than the cycle 4–2–1 cannot exist.

References

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